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Journal of Approximation Theory 134 (2005) 257–266

Journal of Approximation Theory

www.elsevier.com/locate/jat

# A strong convergence theorem for relatively nonexpansive mappings in a Banach space

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Received 10 July 2003; received in revised form 5 January 2005; accepted 16 February 2005

Communicated by Frank Deutsch
Available online 2 April 2005

**Abstract**

In this paper, we prove a strong convergence theorem for relatively nonexpansive mappings in a Banach space by using the hybrid method in mathematical programming. Using this result, we also discuss the problem of strong convergence concerning nonexpansive mappings in a Hilbert space and maximal monotone operators in a Banach space.


*MSC:* primary 47H09; secondary 47H05

*Keywords:* Relatively nonexpansive mapping; Nonexpansive mapping; Asymptotic fixed point; Generalized projection; Maximal monotone operator

## 1. Introduction

Let $E$ be a smooth Banach space and let $E^*$ be the dual of $E$. The function $\phi : E \times E \to \mathbf{R}$ is defined by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx\rangle + \|x\|^2$$

for all $x, y \in E$, where $J$ is the normalized duality mapping from $E$ to $E^*$. Let $C$ be a closed convex subset of $E$, and let $T$ be a mapping from $C$ into itself. We denote by $F(T)$

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0021-9045/$ - see front matter 
doi:10.1016/j.jat.2005.02.007

the set of fixed points of $T$. A point $p$ in $C$ is said to be an asymptotic fixed point of $T$ [13] if $C$ contains a sequence $\{x_n\}$ which converges weakly to $p$ such that the strong $\lim_{n\to\infty}(x_n - Tx_n) = 0$. The set of asymptotic fixed points of $T$ will be denoted by $\hat{F}(T)$. A mapping $T$ from $C$ into itself is called nonexpansive if $\|Tx - Ty\| \leqslant \|x - y\|$ for all $x, y \in C$ and relatively nonexpansive [3–5] if $\hat{F}(T) = F(T)$ and $\phi(p, Tx) \leqslant \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. The asymptotic behavior of a relatively nonexpansive mapping was studied in [3–5]. On the other hand, Nakajo and Takahashi [9] obtained strong convergence theorems for nonexpansive mappings in a Hilbert space. In particular, they studied the strong convergence of the sequence $\{x_n\}$ generated by

$$\begin{cases} x_0 = x \in C, \\ y_n = \alpha_n x_n + (1 - \alpha_n) S x_n, \\ C_n = \{z \in C : \|z - y_n\| \leqslant \|z - x_n\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geqslant 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \; n = 0, 1, 2, \ldots, \end{cases}$$

where $\{\alpha_n\} \subset [0, 1]$, $S$ is a nonexpansive mapping from $C$ into itself and $P_{C_n \cap Q_n}$ is the metric projection from $C$ onto $C_n \cap Q_n$.

Motivated by Nakajo and Takahashi [9], our purpose in this paper is to prove a strong convergence theorem for relatively nonexpansive mappings in a Banach space. Using this result, we also discuss the problem of strong convergence concerning nonexpansive mappings in a Hilbert space and maximal monotone operators in a Banach space.

## 2. Preliminaries

Let $E$ be a real Banach space with norm $\|\cdot\|$ and let $E^*$ be the dual of $E$. Denote by $\langle \cdot, \cdot \rangle$ the duality product. The normalized duality mapping $J$ from $E$ to $E^*$ is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for $x \in E$. When $\{x_n\}$ is a sequence in $E$, we denote strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and weak convergence by $x_n \rightharpoonup x$.

A Banach space $E$ is said to be strictly convex if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is also said to be uniformly convex if $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}, \{y_n\}$ in $E$ such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty} \|\frac{x_n+y_n}{2}\| = 1$. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of $E$. Then the Banach space $E$ is said to be smooth provided

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in U$. It is also said to be uniformly smooth if the limit is attained uniformly for $x, y \in U$. It is well known that if $E$ is smooth, then the duality mapping $J$ is single valued. It is also known that if $E$ is uniformly smooth, then $J$ is uniformly norm-to-norm continuous on each bounded subset of $E$. Some properties of the duality mapping have been given in [6,12,16,17]. A Banach space $E$ is said to have the Kadec–Klee property if a sequence $\{x_n\}$ of $E$ satisfying that $x_n \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$, then $x_n \to x$. It is known that if $E$ is uniformly convex, then $E$ has the Kadec–Klee property; see [6,16,17] for more details. Let $E$ be a smooth Banach space. The function $\phi : E \times E \to \mathbf{R}$ is defined by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx\rangle + \|x\|^2$$

for $x, y \in E$. It is obvious from the definition of the function $\phi$ that

$$(\|y\| - \|x\|)^2 \leqslant \phi(y, x) \leqslant (\|y\| + \|x\|)^2 \tag{2.1}$$

for all $x, y \in E$.

**Remark 2.1.** If $E$ is a strictly convex and smooth Banach space, then for $x, y \in E$, $\phi(y, x) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(y, x) = 0$ then $x = y$. From (2.1), we have $\|x\| = \|y\|$. This implies $\langle y, Jx\rangle = \|y\|^2 = \|Jx\|^2$. From the definition of $J$, we have $Jx = Jy$. Since $J$ is one-to-one, we have $x = y$; see [6,16,17] for more details.

Recently, Kamimura and Takahashi [7] proved the following result. This plays an important role in the proof of the main theorem.

**Proposition 2.1** (*Kamimura and Takahashi [7]*). *Let $E$ be a uniformly convex and smooth Banach space and let $\{y_n\}, \{z_n\}$ be two sequences of $E$. If $\phi(y_n, z_n) \to 0$ and either $\{y_n\}$ or $\{z_n\}$ is bounded, then $y_n - z_n \to 0$.*

Let $C$ be a nonempty closed convex subset of $E$. Suppose that $E$ is reflexive, strictly convex and smooth. Then, for any $x \in E$, there exists a point $x_0 \in C$ such that

$$\phi(x_0, x) = \min_{y \in C} \phi(y, x).$$

The mapping $P_C : E \to C$ defined by $P_C x = x_0$ is called the generalized projection [1,2,7]. The following are well-known results. For example, see [1,2,7]

**Proposition 2.2** (*Alber [1], Alber and Reich [2], Kamimura and Takahashi [7]*). *Let $C$ be a nonempty closed convex subset of a smooth Banach space $E$ and $x \in E$. Then, $x_0 = P_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0\rangle \geqslant 0$$

*for $y \in C$.*

**Proposition 2.3** (*Alber [1], Alber and Reich [2], Kamimura and Takahashi [7]*). *Let $E$ be a reflexive, strictly convex and smooth Banach space, let $C$ be a nonempty closed convex subset of $E$ and let $x \in E$. Then*

$$\phi(y, P_C x) + \phi(P_C x, x) \leqslant \phi(y, x)$$

*for all $y \in C$.*

Concerning the set of fixed points of a relatively nonexpansive mapping, we can prove the following result.

**Proposition 2.4.** *Let E be a strictly convex and smooth Banach space, let C be a closed convex subset of E, and let T be a relatively nonexpansive mapping from C into itself. Then* $F(T)$ *is closed and convex.*

**Proof.** We first show that $F(T)$ is closed. Let $\{x_n\}$ be a sequence of $F(T)$ such that $x_n \to \hat{x} \in C$. From the definition of $T$,

$$\phi(x_n, T\hat{x}) \leqslant \phi(x_n, \hat{x})$$

for each $n \in \mathbf{N}$. This implies,

$$\phi(\hat{x}, T\hat{x}) = \lim_{n\to\infty} \phi(x_n, T\hat{x}) \leqslant \lim_{n\to\infty} \phi(x_n, \hat{x}) = \phi(\hat{x}, \hat{x}) = 0.$$

Therefore we obtain $\hat{x} = T\hat{x}$. So, we have $\hat{x} \in F(T)$. Next, we show that $F(T)$ is convex. For $x, y \in F(T)$ and $t \in (0, 1)$, put $z = tx + (1-t)y$. It is sufficient to show $Tz = z$. In fact, we have

$$\begin{aligned}
\phi(z, Tz) &= \|z\|^2 - 2\langle z, JTz\rangle + \|Tz\|^2 \\
&= \|z\|^2 - 2\langle tx + (1-t)y, JTz\rangle + \|Tz\|^2 \\
&= \|z\|^2 - 2t\langle x, JTz\rangle - 2(1-t)\langle y, JTz\rangle + \|Tz\|^2 \\
&= \|z\|^2 + t\phi(x, Tz) + (1-t)\phi(y, Tz) - t\|x\|^2 - (1-t)\|y\|^2 \\
&\leqslant \|z\|^2 + t\phi(x, z) + (1-t)\phi(y, z) - t\|x\|^2 - (1-t)\|y\|^2 \\
&= \|z\|^2 - 2\langle tx + (1-t)y, Jz\rangle + \|z\|^2 \\
&= \|z\|^2 - 2\langle z, Jz\rangle + \|z\|^2 = 0.
\end{aligned}$$

This implies $z = Tz$. □

## 3. Main result

Now, we can prove a strong convergence theorem for relatively nonexpansive mappings in a Banach space by using the hybrid method in mathematical programming.

**Theorem 3.1.** *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E, let T be a relatively nonexpansive mapping from C into itself, and let* $\{\alpha_n\}$ *be a sequence of real numbers such that* $0 \leqslant \alpha_n < 1$ *and* $\limsup_{n\to\infty} \alpha_n < 1$. *Suppose that* $\{x_n\}$ *is given by*

$$\begin{cases}
x_0 = x \in C, \\
y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\
H_n = \{z \in C : \phi(z, y_n) \leqslant \phi(z, x_n)\}, \\
W_n = \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \geqslant 0\}, \\
x_{n+1} = P_{H_n \cap W_n} x, \; n = 0, 1, 2, \ldots,
\end{cases}$$

*where J is the duality mapping on E. If* $F(T)$ *is nonempty, then* $\{x_n\}$ *converges strongly to* $P_{F(T)}x$, *where* $P_{F(T)}$ *is the generalized projection from C onto* $F(T)$.

**Proof.** We first show that $H_n$ and $W_n$ are closed and convex for each $n \in \mathbf{N} \cup \{0\}$. From the definition of $H_n$ and $W_n$, it is obvious that $H_n$ is closed and $W_n$ is closed and convex for each $n \in \mathbf{N} \cup \{0\}$. We show that $H_n$ is convex. Since $\phi(z, y_n) \leqslant \phi(z, x_n)$ is equivalent to

$$2\langle z, Jx_n - Jy_n \rangle + \|y_n\|^2 - \|x_n\|^2 \leqslant 0,$$

it follows that $H_n$ is convex.

Next, we show that $F(T) \subset H_n \cap W_n$ for each $n \in \mathbf{N} \cup \{0\}$. Let $u \in F(T)$ and let $n \in \mathbf{N} \cup \{0\}$. Then from

$$\begin{aligned}
\phi(u, y_n) &= \phi(u, J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n)) \\
&= \|u\|^2 - 2\langle u, \alpha_n Jx_n + (1-\alpha_n)JTx_n \rangle + \|\alpha_n Jx_n + (1-\alpha_n)JTx_n\|^2 \\
&\leqslant \|u\|^2 - 2\alpha_n \langle u, Jx_n \rangle - 2(1-\alpha_n)\langle u, JTx_n \rangle + \alpha_n \|x_n\|^2 \\
&\quad + (1-\alpha_n)\|Tx_n\|^2 \\
&= \alpha_n(\|u\|^2 - 2\langle u, Jx_n \rangle + \|x_n\|^2) + (1-\alpha_n)(\|u\|^2 - 2\langle u, JTx_n \rangle \\
&\quad + \|Tx_n\|^2) \\
&= \alpha_n \phi(u, x_n) + (1-\alpha_n)\phi(u, Tx_n) \\
&\leqslant \alpha_n \phi(u, x_n) + (1-\alpha_n)\phi(u, x_n) \\
&= \phi(u, x_n),
\end{aligned}$$

we have $u \in H_n$. Therefore we obtain

$$F(T) \subset H_n$$

for each $n \in \mathbf{N} \cup \{0\}$. On the other hand, it is clear that $F(T) \subset H_0 \cap W_0$. Suppose that $F(T) \subset H_k \cap W_k$ for some $k \in \mathbf{N}$. There exists an element $x_{k+1} \in H_k \cap W_k$ such that $x_{k+1} = P_{H_k \cap W_k} x$. From Proposition 2.2, there holds

$$\langle x_{k+1} - z, Jx - Jx_{k+1} \rangle \geqslant 0$$

for each $z \in H_k \cap W_k$. Since $F(T) \subset H_k \cap W_k$, we have $\langle x_{k+1} - u, Jx - Jx_{k+1} \rangle \geqslant 0$ for every $u \in F(T)$ and hence $F(T) \subset W_{k+1}$. Therefore we have $F(T) \subset H_{k+1} \cap W_{k+1}$. This implies that $\{x_n\}$ is well defined. It follows from the definition of $W_n$ and Proposition 2.2 that $x_n = P_{W_n} x$. Using $x_n = P_{W_n} x$ and Proposition 2.3, we have

$$\phi(x_n, x) = \phi(P_{W_n} x, x) \leqslant \phi(u, x) - \phi(u, x_n) \leqslant \phi(u, x)$$

for each $u \in F(T) \subset W_n$ for each $n \in \mathbf{N} \cup \{0\}$. Therefore, $\phi(x_n, x)$ is bounded. Moreover, from (2.1), we have that $\{x_n\}$ is bounded.

Since $x_{n+1} = P_{H_n \cap W_n} x \in W_n$ and Proposition 2.3, we have

$$\phi(x_n, x) \leqslant \phi(x_{n+1}, x)$$

for each $n \in \mathbf{N} \cup \{0\}$. Therefore $\{\phi(x_n, x)\}$ is nondecreasing. So there exists the limit of $\phi(x_n, x)$. From Proposition 2.3, we have

$$\begin{aligned}
\phi(x_{n+1}, x_n) &= \phi(x_{n+1}, P_{W_n} x) \leqslant \phi(x_{n+1}, x) - \phi(P_{W_n} x, x) \\
&= \phi(x_{n+1}, x) - \phi(x_n, x)
\end{aligned}$$

for each $n \in \mathbf{N} \cup \{0\}$. This implies that $\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0$. Since $x_{n+1} = P_{H_n \cap W_n} x \in H_n$, from the definition of $H_n$, we also have

$$\phi(x_{n+1}, y_n) \leqslant \phi(x_{n+1}, x_n)$$

for each $n \in \mathbf{N} \cup \{0\}$. Tending $n \to \infty$, we have $\lim_{n\to\infty} \phi(x_{n+1}, y_n) = 0$. Using Proposition 2.1, we obtain

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = \lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Since $J$ is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|Jx_{n+1} - Jy_n\| = \lim_{n\to\infty} \|Jx_{n+1} - Jx_n\| = 0. \tag{3.1}$$

On the other hand, we have, for each $n \in \mathbf{N} \cup \{0\}$,

$$\begin{aligned}
\|Jx_{n+1} - Jy_n\| &= \|Jx_{n+1} - (\alpha_n Jx_n + (1-\alpha_n)JTx_n))\| \\
&= \|\alpha_n(Jx_{n+1} - Jx_n) + (1-\alpha_n)(Jx_{n+1} - JTx_n)\| \\
&= \|(1-\alpha_n)(Jx_{n+1} - JTx_n) - \alpha_n(Jx_n - Jx_{n+1})\| \\
&\geqslant (1-\alpha_n)\|Jx_{n+1} - JTx_n\| - \alpha_n\|Jx_n - Jx_{n+1}\|
\end{aligned}$$

and hence

$$\begin{aligned}
\|Jx_{n+1} - JTx_n\| &\leqslant \frac{1}{1-\alpha_n}(\|Jx_{n+1} - Jy_n\| + \alpha_n\|Jx_n - Jx_{n+1}\|) \\
&\leqslant \frac{1}{1-\alpha_n}(\|Jx_{n+1} - Jy_n\| + \|Jx_n - Jx_{n+1}\|).
\end{aligned}$$

From (3.1) and $\limsup_{n\to\infty} \alpha_n < 1$, we obtain

$$\lim_{n\to\infty} \|Jx_{n+1} - JTx_n\| = 0.$$

Since $J^{-1}$ is also uniformly norm-to-norm continuous on bounded sets, we obtain

$$\lim_{n\to\infty} \|x_{n+1} - Tx_n\| = \lim_{n\to\infty} \|J^{-1}(Jx_{n+1}) - J^{-1}(JTx_n)\| = 0.$$

From

$$\begin{aligned}
\|x_n - Tx_n\| &= \|x_n - x_{n+1} + x_{n+1} - Tx_n\| \\
&\leqslant \|x_n - x_{n+1}\| + \|x_{n+1} - Tx_n\|,
\end{aligned}$$

we have $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Therefore, if $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \hat{x} \in C$, then $\hat{x} \in \hat{F}(T) = F(T)$.

Finally, we show that $x_n \to P_{F(T)}x$. Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \hat{x} \in F(T)$ and $w = P_{F(T)}x$. For any $n \in \mathbf{N}$, from $x_{n+1} = P_{H_n \cap W_n} x$ and $w \in F(T) \subset H_n \cap W_n$, we have $\phi(x_{n+1}, x) \leqslant \phi(w, x)$. On the other hand, from weakly lower semicontinuity of the norm, we have

$$\begin{aligned}
\phi(\hat{x}, x) &= \|\hat{x}\|^2 - 2\langle \hat{x}, Jx\rangle + \|x\|^2 \\
&\leqslant \liminf_{k\to\infty} (\|x_{n_k}\|^2 - 2\langle x_{n_k}, Jx\rangle + \|x\|^2)
\end{aligned}$$

$$
\begin{aligned}
&= \liminf_{k\to\infty} \phi(x_{n_k}, x) \\
&\leqslant \limsup_{k\to\infty} \phi(x_{n_k}, x) \\
&\leqslant \phi(w, x).
\end{aligned}
$$

From the definition of $P_{F(T)}x$, we obtain $\hat{x} = w$ and hence $\lim_{k\to\infty} \phi(x_{n_k}, x) = \phi(w, x)$. So, we have

$$\lim_{k\to\infty} \|x_{n_k}\| = \|w\|.$$

Using the Kadec–Klee property of $E$, we obtain that $\{x_{n_k}\}$ converges strongly to $P_{F(T)}x$. Since $\{x_{n_k}\}$ is an arbitrary weakly convergent sequence of $\{x_n\}$, we can conclude that $\{x_n\}$ converges strongly to $P_{F(T)}x$. $\square$

## 4. Applications

In this section, we discuss the problem of strong convergence concerning nonexpansive mappings in a Hilbert space and maximal monotone operators in a Banach space. Using Theorem 3.1, we first obtain the result of [9].

**Theorem 4.1** (*Nakajo and Takahashi[9]*). *Let $C$ be a nonempty closed convex subset of a Hilbert space $H$, and let $T$ be a nonexpansive mapping of $C$ into itself such that $F(T)$ is nonempty. Suppose that $\{x_n\}$ is given by*

$$
\begin{cases}
x_0 = x \in C, \\
y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\
C_n = \{z \in C : \|z - y_n\| \leqslant \|z - x_n\|\}, \\
Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \geqslant 0\}, \\
x_{n+1} = P_{C_n \cap Q_n} x, \ n = 0, 1, 2, \ldots,
\end{cases}
$$

*where $\{\alpha_n\} \subset [0, a)$ for some $a \in [0, 1)$ and $P_{C_n\cap Q_n}$ is the metric projection from $C$ onto $C_n \cap Q_n$. Then $\{x_n\}$ converges strongly to $P_{F(T)}x$, where $P_{F(T)}$ is the metric projection from $C$ onto $F(T)$.*

**Proof.** It is sufficient to prove that if $T$ is nonexpansive, then $T$ is relatively nonexpansive. It is obvious that $F(T) \subset \hat{F}(T)$. If $u \in \hat{F}(T)$, then there exists $\{x_n\} \subset C$ such that $x_n \rightharpoonup u$ and $x_n - Tx_n \to 0$. Since $T$ is nonexpansive, $T$ is demiclosed. So, we have $u = Tu$. This implies $F(T) = \hat{F}(T)$. Further, in a Hilbert space $H$, we know that

$$\phi(x, y) = \|x - y\|^2$$

for every $x, y \in H$. So, $\|Tx - Ty\| \leqslant \|x - y\|$ is equivalent to $\phi(Tx, Ty) \leqslant \phi(x, y)$. Therefore, $T$ is relatively nonexpansive. Using Theorem 3.1, we obtain the desired result. $\square$

Let $A$ be a multivalued operator from $E$ to $E^*$ with domain $D(A) = \{z \in E : Az \neq \phi\}$ and range $R(A) = \cup\{Az : z \in D(A)\}$. An operator $A$ is said to be monotone if

$\langle x_1 - x_2, y_1 - y_2\rangle \geqslant 0$ for each $x_i \in D(A)$ and $y_i \in Tx_i$, $i = 1, 2$. A monotone operator $A$ is said to be maximal if its graph $G(A) = \{(x, y) : y \in Ax\}$ is not properly contained in the graph of any other monotone operator. We know that if $A$ is a maximal monotone operator, then $A^{-1}0$ is closed and convex. The following result is also well-known.

**Theorem 4.2** (*Rockafellar [14]*). *Let $E$ be a reflexive, strictly convex and smooth Banach space and let $A$ be a monotone operator from $E$ to $E^*$. Then $A$ is maximal if and only if $R(J + rA) = E^*$ for all $r > 0$.*

Let $E$ be a reflexive, strictly convex and smooth Banach space, and let $A$ be a maximal monotone operator from $E$ to $E^*$. Using Theorem 4.2 and strict convexity of $E$, we obtain that for every $r > 0$ and $x \in E$, there exists a unique $x_r \in D(A)$ such that

$$Jx \in Jx_r + rAx_r.$$

If $J_r x = x_r$, then we can define a single valued mapping $J_r : E \to D(A)$ by $J_r = (J + rA)^{-1}J$ and such a $J_r$ is called the resolvent of $A$. We know that $A^{-1}0 = F(J_r)$ for all $r > 0$; see[16,17] for more details. Using Theorem 3.1, we can consider the problem of strong convergence concerning maximal monotone operators in a Banach space. Such a problem has been also studied in [7–11,13,15].

**Theorem 4.3.** *Let $E$ be a uniformly convex and uniformly smooth Banach space, let $A$ be a maximal monotone operator from $E$ to $E^*$, let $J_r$ be a resolvent of $A$, where $r > 0$ and let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leqslant \alpha_n < 1$ and $\limsup_{n\to\infty} \alpha_n < 1$. Suppose that $\{x_n\}$ is given by*

$$\begin{cases} x_0 = x \in E, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JJ_r x_n), \\ H_n = \{z \in E : \phi(z, y_n) \leqslant \phi(z, x_n)\}, \\ W_n = \{z \in E : \langle x_n - z, Jx - Jx_n\rangle \geqslant 0\}, \\ x_{n+1} = P_{H_n \cap W_n} x, \ n = 0, 1, 2, \ldots, \end{cases}$$

*where $J$ is the duality mapping on $E$. If $A^{-1}0$ is nonempty, then $\{x_n\}$ converges strongly to $P_{A^{-1}0}x$ where $P_{A^{-1}0}$ is the generalized projection from $E$ onto $A^{-1}0$.*

**Proof.** We first show that $\hat{F}(J_r) \subset A^{-1}0$. Let $p \in \hat{F}(J_r)$. Then, there exists $\{z_n\} \subset E$ such that $z_n \rightharpoonup p$ and $\lim_{n\to\infty}(z_n - J_r z_n) = 0$. Since $J$ is uniformly norm-to-norm continuous on bounded sets, we obtain

$$\frac{1}{r}(Jz_n - JJ_r z_n) \to 0.$$

It follows from $\frac{1}{r}(Jz_n - JJ_r z_n) \in AJ_r z_n$ and the monotonicity of $A$ that

$$\langle w - J_r z_n, w^* - \tfrac{1}{r}(Jz_n - JJ_r z_n)\rangle \geqslant 0$$

for all $w \in D(A)$ and $w^* \in Aw$. Letting $n \to \infty$, we have

$$\langle w - p, w^*\rangle \geqslant 0$$

for all $w \in D(A)$ and $w^* \in Aw$. Therefore from the maximality of $A$, we obtain $p \in A^{-1}0$. On the other hand, we know that $F(J_r) = A^{-1}0$ and $F(J_r) \subset \hat{F}(J_r)$, therefore $A^{-1}0 = F(J_r) = \hat{F}(J_r)$. Next we show that $J_r$ is a relatively nonexpansive mapping with respect to $A^{-1}0$. Let $w \in E$ and $p \in A^{-1}0$. From the monotonicity of $A$, we have

$$\begin{aligned}
\phi(p, J_r w) &= \|p\|^2 - 2\langle p, JJ_r w\rangle + \|J_r w\|^2 \\
&= \|p\|^2 + 2\langle p, Jw - JJ_r w - Jw\rangle + \|J_r w\|^2 \\
&= \|p\|^2 + 2\langle p, Jw - JJ_r w\rangle - 2\langle p, Jw\rangle + \|J_r w\|^2 \\
&= \|p\|^2 - 2\langle J_r w - p - J_r w, Jw - JJ_r w\rangle - 2\langle p, Jw\rangle + \|J_r w\|^2 \\
&= \|p\|^2 - 2\langle J_r w - p, Jw - JJ_r w\rangle \\
&\quad +2\langle J_r w, Jw - JJ_r w\rangle - 2\langle p, Jw\rangle + \|J_r w\|^2 \\
&= \|p\|^2 - 2r\langle J_r w - p, \tfrac{1}{r}(Jw - JJ_r w)\rangle \\
&\quad +2\langle J_r w, Jw - JJ_r w\rangle - 2\langle p, Jw\rangle + \|J_r w\|^2 \\
&\leqslant \|p\|^2 + 2\langle J_r w, Jw - JJ_r w\rangle - 2\langle p, Jw\rangle + \|J_r w\|^2 \\
&= \|p\|^2 - 2\langle p, Jw\rangle + \|w\|^2 - \|J_r w\|^2 + 2\langle J_r w, Jw\rangle - \|w\|^2 \\
&= \phi(p, w) - \phi(J_r w, w) \\
&\leqslant \phi(p, w).
\end{aligned}$$

This implies that $J_r$ is a relatively nonexpansive mapping. Using Theorem 3.1, we can conclude that $\{x_n\}$ converges strongly to $P_{A^{-1}0}x$. □

## Acknowledgments

The authors would like to thank the referee for valuable suggestions which helped to improve this manuscript.

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